

**MONTE CARLO METHODS  
FOR HIGH ENERGY PHYSICS**

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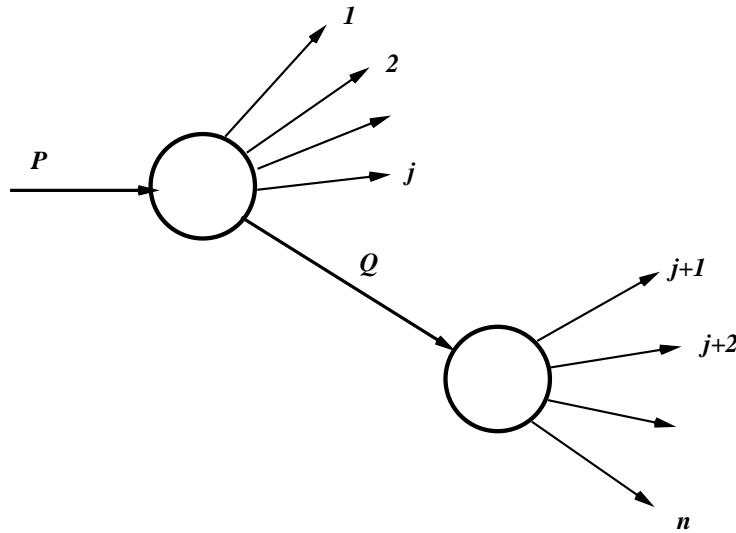
**Lecture 4**

Slides, program sources: <http://home.cern.ch/jadach>

## Outline:

1. Cascade parametrization of the phase space (for multibranching MC's).
2. Lorentz group parametrization of the phase space.

Most common parametrization of the phase space in MC's



Phase space parametrization in most of the MC's is based on the “cascade identity”.

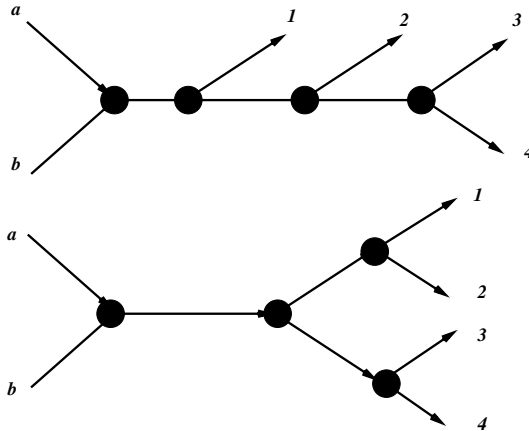
Defining LORENTZ INVARIANT PHASE SPACE as:

$$d\tau_n(P; p_1, p_2, \dots, p_n) \equiv \delta^{(4)}(P - \sum_{j=1}^n p_j) \prod_{j=1}^n \frac{d^3 p_j}{2p_j^0}$$

the identity is:

$$\begin{aligned} \int d\tau_n(P; p_1, p_2, \dots, p_n) &= \\ &= \int ds_Q \int d\tau_j(P; p_1, p_2, \dots, p_j, Q) \int d\tau_{n-j}(Q; p_{j+1}, j+2, \dots, p_n) \end{aligned}$$

Most common cascade parametrization of the phase space in MC's



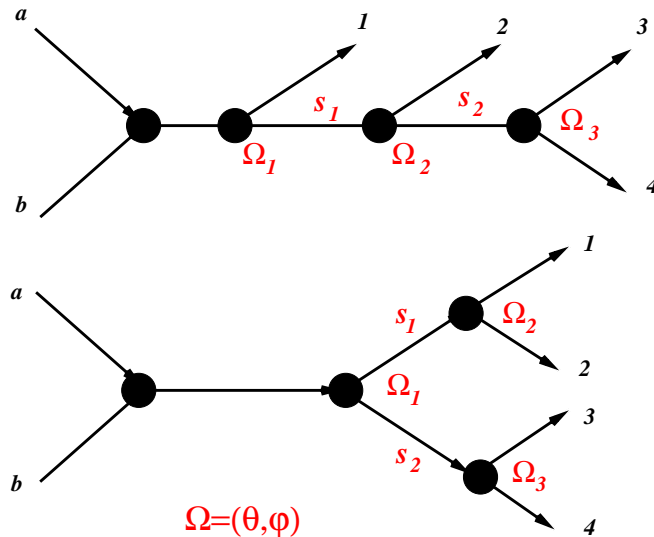
Employing cascade identity several times we may always decompose  $n$ -body phase space into cascade of the 2-body phase spaces.

For example in case of 4 final particles there are 2 nontrivial cascades (kinematic trees) and about 50 permutations altogether.

(This is what is in KORALW MC or EXCALIBUR type programs.)

Note that even for multiperipheral diagram the trivial tree is adequate (we can find relation between angle and the transfer, Kajantie Byckling 1965).

### Cascade parametrization of the phase space in MC's



In this case phase space is parametrized in terms of 2 masses and 6 angles. For each tree (including permutations) one may know from Feynman diagram possible menu of singularities for effective masses  $1, 1/s, 1/((s - M^2)^2 + M^2\Gamma^2)\dots$  and for  $\theta$  angle  $1, 1/t, 1/t^2, 1/(t - M^2)\dots$

This leads to many branches in the MC. Each of them represents one kinematical tree and one assignment of singularities.

Thanks to “slight simplifications” countered by the correcting weights we can generate  $(s_1, s_2, \Omega_1, \Omega_2, \Omega_3)$  by analytical mapping.

This is how it looks in many MC programs. [But not in all of them, see next slides...](#)

**Lorentz group parametrization of the phase space**

There is an interesting and powerful extension of the cascade parametrization, which relies strictly on the “rigid body” parametrization on the group of the four momenta.

It allows to “attach the Lorentz frame” to a group of momenta.

These momenta can be timelike (s-channel) or spacelike (t-channel).

This technique is particularly useful if we want to have 100% control of spin polarization effects for the multibody final state, because it precisely keeps track of all “azimuthal angles” which influence the phases of the helicity amplitudes.

In the following I shall “give the taste” of what it is.

Only s-channel version is shown here

(t-channel version is a bit more complicated.)

This s-channel identity was employed for the first time in the YFS3 MC event generator in 1988/89. (t-channel version in BHLUMI).

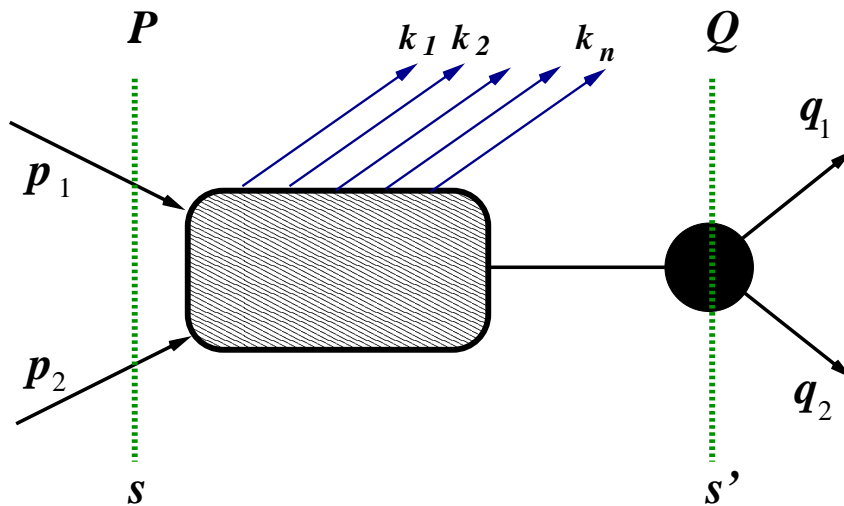
## Lorentz group parametrization of the phase space

$$\sigma_n = \int \tau_{n+2}(p_1 + p_2; k_1, \dots, k_n, q_1, q_2) \Sigma_n(p_i, q_j, k_l)$$

Lets introduce several “spurious” integrations countered by delta functions:

$$\sigma_n = \int d^4 P \int ds' \int \prod_{j=1}^n \frac{d^3 k_j}{2k_j^0} \int d^4 Q \int \frac{d^3 q_1}{2q_1^0} \frac{d^3 q_2}{2q_2^0} \delta^{(3)}(P) \delta(s - P^2)$$

$$\delta^{(4)}\left(P - Q - \sum_{j=1}^n k_j\right) \delta^{(4)}(Q - q_1 - q_2) \delta(Q^2 - s') 2s^{1/2} \Sigma_n(p_i, q_j, k_l)$$



**Lets go!**

Even more integrations and delta's are added implantiing the identity:

$$\int d \cos \omega d \psi 2 \delta \left( \frac{X^2}{s_Q} - 1 \right) \frac{d^4 X}{s_Q^2} \delta^{(3)} \left( L_A^{-1} \frac{Q}{\sqrt{s_Q}} \right) \delta^{\{12\}} \left( L_A^{-1} \frac{q_1}{|\vec{q}_1|_Q} \right) \equiv 1.$$

where the big transformation from the rests frame of  $Q = q_1 + q_2$  to LAB enters:

$$L_A = R(\psi, \omega) B_X = B_{RX} R, \quad R = R(\psi, \omega) = R_3(\psi) R_2(\omega)$$

$$L_A^{-1} = B_X^{-1} R^{-1}(\psi, \omega) = R^{-1} B_{RX}^{-1}$$

After that the integral looks as follows (all variables are integration variables!):

$$\begin{aligned} \sigma_n = & \int ds' \int d \cos \omega d \psi 2 \delta \left( \frac{X^2}{s'} - 1 \right) \frac{d^4 X}{s'^2} \\ & \int d^4 P \int \prod_{j=1}^n \frac{d^3 k_j}{2k_j^0} \int d^4 Q \int \frac{d^3 q_1}{2q_1^0} \frac{d^3 q_2}{2q_2^0} \delta^{(3)}(P) \delta(s - P^2) \\ & \delta^{(4)} \left( P - Q - \sum_{j=1}^n k_j \right) \delta^{(4)}(Q - q_1 - q_2) \delta(Q^2 - s') \\ & \delta^{(3)} \left( L_A^{-1} \frac{Q}{\sqrt{s_Q}} \right) \delta^{\{12\}} \left( L_A^{-1} \frac{q_1}{|\vec{q}_1|_Q} \right) 2s^{1/2} \Sigma_n(p_i, q_j, k_l) \end{aligned}$$

Here  $R_i(x)$  are matrices of rotations around  $i$  - th axis and  $B_X$  is matrix of the “parallel boost” transformation defined by timelike vector  $X$ , see next slide.

### The Parallel Lorentz boost

from to the rest frame of the timelike 4-momentum  $q = (q^0, \vec{q})$ ,  $q^2 = m^2$  to LAB:

$$B_q \equiv \begin{bmatrix} \frac{q^0}{m}, & \frac{\vec{q}^T}{m} \\ \frac{\vec{q}}{m}, & I + \frac{\vec{q} \otimes \vec{q}}{m(m+q^0)} \end{bmatrix},$$

where  $T$  marks matrix transposition and  $\otimes$  tensor product.

Basic properties and relations:

$$B_q \hat{q} = q, \quad \hat{q} \equiv (m, 0, 0, 0),$$

$$B_{B_3(\eta)\hat{q}} = B_3(\eta),$$

$$B_q^{-1} = B_{q^P}, \quad q^P = (q^0, -\vec{q}),$$

$$RB_q R^{-1} = RB_q R^T = B_{Rq},$$

$$RB_q = B_{Rq} R, \quad \text{where } R \text{ is any 3-dimensional rotation.}$$

For  $q = R\hat{q}_z = R_3(\phi)R_2(\theta)q_z$ ;  $q_z = (q^0, 0, 0, |\vec{q}|) = B_3(\eta)\hat{q}$ , we have:

$$B_q = B_{Rq_z} = RB_{q_z} R^{-1} = R_3(\phi)R_2(\theta)B_3(\eta)R_2(-\theta)R_3(-\phi)$$

### The Parallel Lorentz boost

The following identity holds:

$$\int 2\delta\left(\frac{x^2}{m^2} - 1\right) \frac{d^4x}{m^4} \delta^{(3)}\left(L_x^{-1} \frac{p}{m}\right) f(x, q) = f(p, q)$$

Proof:

$$\begin{aligned} & \int \frac{2}{m^2} \delta(x^2 - m^2) d^4x \delta^{(3)}\left(L_x^{-1} \frac{p}{m}\right) f(x, q) \\ &= \int \sinh^2 \eta_x d\eta_x d\cos\theta_x d\phi_x \delta^{(3)}\left(L_x^{-1} \frac{p}{m}\right) f(x, q) \\ &= \int \sinh^2 \eta_x d\eta_x d\cos\theta_x d\phi_x \delta^{(3)}\left(B_3(-\eta_x) R_3(\phi_x) R_2(-\theta_x) \frac{p}{m}\right) f(x, q) \\ &= f(\hat{x}(p), q) = f(p, q). \end{aligned}$$

### Change of variables

Now we change of variables in the integral which is just identical to the Lorentz transformation  $L_A$ ; it formally looks as follows:

$$P \rightarrow L_A P, Q \rightarrow L_A Q, k_j \rightarrow L_A k_j, q_i \rightarrow L_A q_i,$$

First we do a substitution  $p = L_A p'$ ; the new variables are  $p' = L_A^{-1} p$ . Next we rename  $p' \rightarrow p$ . All parameters in  $L_A$  are passive variables of the outer integrations.

In most places  $L_A$  drops out because of Lorentz covariance:

$$\begin{aligned} \sigma_n &= \int ds' \int d \cos \omega d\psi \ 2\delta\left(\frac{X^2}{s'} - 1\right) \frac{d^4 X}{s'^2} \\ &\int d^4 P \int \prod_{j=1}^n \frac{d^3 k_j}{2k_j^0} \int d^4 Q \int \frac{d^3 q_1}{2q_1^0} \frac{d^3 q_2}{2q_2^0} \delta^{(3)}(L_A P) \delta(s - P^2) \\ &\delta^{(4)}\left(P - Q - \sum_{j=1}^n k_j\right) \delta^{(4)}(Q - q_1 - q_2) \delta(Q^2 - s') \\ &\delta^{(3)}\left(\frac{q_1 + q_2}{\sqrt{s'}}\right) \delta^{\{12\}}\left(\frac{q_1}{|\vec{q}_1|_Q}\right) 2s^{1/2} \Sigma_n(p_i, q_j, k_l) \end{aligned}$$

Now we reduce most of  $\delta$ -functions starting with the easiest...

Momenta  $q_1$  and  $q_2$  are fixed!

**First we apply identity:**

$$\int d^4 Q \delta(Q^2 - s') \int \frac{d^3 q_1}{2q_1^0} \frac{d^3 q_2}{2q_2^0} \delta^{(3)}\left(\frac{q_1+q_2}{\sqrt{s'}}\right) \delta^{(4)}(Q - q_1 - q_2) \delta^{\{12\}}\left(\frac{q_1}{|\vec{q}_1|_Q}\right) = s' \frac{1}{2} \beta(s'; q_1^2, q_2^2),$$

**to our integral:**

$$\begin{aligned} \sigma_n &= \int ds' \int d \cos \omega d\psi \ 2\delta\left(\frac{X^2}{s'} - 1\right) \frac{d^4 X}{s'^2} \\ &\int d^4 P \int \prod_{j=1}^n \frac{d^3 k_j}{2k_j^0} \int d^4 Q \int \frac{d^3 q_1}{2q_1^0} \frac{d^3 q_2}{2q_2^0} \delta^{(3)}(L_A P) \delta(s - P^2) \\ &\delta^{(4)}\left(P - Q - \sum_{j=1}^n k_j\right) \delta^{(4)}(Q - q_1 - q_2) \delta(Q^2 - s') \\ &\delta^{(3)}\left(\frac{q_1+q_2}{\sqrt{s'}}\right) \delta^{\{12\}}\left(\frac{q_1}{|\vec{q}_1|_Q}\right) 2s^{1/2} \Sigma_n(p_i, q_j, k_l) \end{aligned}$$

**where**

$$q_1 = \hat{q}_1 = \left(0, 0, |\vec{q}_1|_Q, \sqrt{s'}/2\right), \quad q_2 = \hat{q}_2 = \left(0, 0, -|\vec{q}_1|_Q, \sqrt{s'}/2\right)$$

$$\vec{q}_1|_Q = \frac{\sqrt{s'}}{2} \beta(s'; q_1^2, q_2^2) = \frac{\sqrt{s'}}{2} \beta_f.$$

**are the solutions of constraints imposed by the reduced  $\delta$ 's!!!**

**The net result of the first reduction is:**

$$\sigma_n = \int d^4 P \int ds' \int d \cos \omega d\psi 2\delta\left(\frac{X^2}{s'} - 1\right) \frac{d^4 X}{s'^2} \int \prod_{j=1}^n \frac{d^3 k_j}{2k_j^0}$$

$$\delta^{(3)}(L_A P) \delta(s - P^2) \delta^{(4)}\left(P - \hat{Q} - \sum_{j=1}^n k_j\right) 2s^{1/2} \frac{s' \beta_f}{16} \Sigma_n(p_i, q_j, k_l) \text{ is}$$

**exactly what we wanted, because at this point  $\hat{q}_i$  are not the integration variables, they are fixed as the constant vectors depending only on  $s'$ !**

**We denote:**  $\hat{Q} \equiv \hat{q}_1 + \hat{q}_1 = (0, 0, 0, \sqrt{s'})$ .

**The next step is rather trivial – we simply apply:**

$$\int d^4 P \delta(s - P^2) \delta^{(4)}\left(P - \hat{Q} - \sum_{j=1}^n k_j\right) = \delta(s - (\hat{Q} + \sum_{j=1}^n k_j)^2)$$

**arriving to**

$$\sigma_n = \int ds' \int d \cos \omega d\psi \int \prod_{j=1}^n \frac{d^3 k_j}{2k_j^0} 2\delta\left(\frac{X^2}{s'} - 1\right) \frac{d^4 X}{s'^2}$$

$$\delta^{(3)}(L_A(\hat{Q} + \sum_{j=1}^n k_j)) \delta(s - (\hat{Q} + \sum_{j=1}^n k_j)^2) \frac{s' \beta_f}{8} s^{1/2} \Sigma_n(p_i, q_j, k_l).$$

**Here comes more interesting reduction:**

$$\int 2\delta\left(\frac{X^2}{s'} - 1\right) \frac{d^4 X}{s'^2} \delta^{(3)}(L_A \hat{P}) = \int 2\delta\left(\frac{X^2}{s'} - 1\right) \frac{d^4 X}{s'^2} \delta^{(3)}((B_{XP})^{-1} \hat{P}) \equiv s^{-3/2},$$

**and  $\hat{P} = \hat{Q} + \sum_{j=1}^n k_j$  is initial 4-mom., as seen from rest frame of  $Q$ .**

**In the process of reducing  $\delta$ 's parameters in  $L_A$  got completely defined:**

$$k_i|_{CMS} = L_A k_i, \quad q_i|_{CMS} = L_A \hat{q}_i,$$

$$L_A = R_3(\psi) R_2(\omega) B_{\hat{P}P} = R_3(\psi) R_2(\omega) B_{\hat{P}}^{-1}.$$

## Lorentz group parametrization

Final results is very simple. Jacobian is, however, reminding us of the fact that we did something non-trivial:

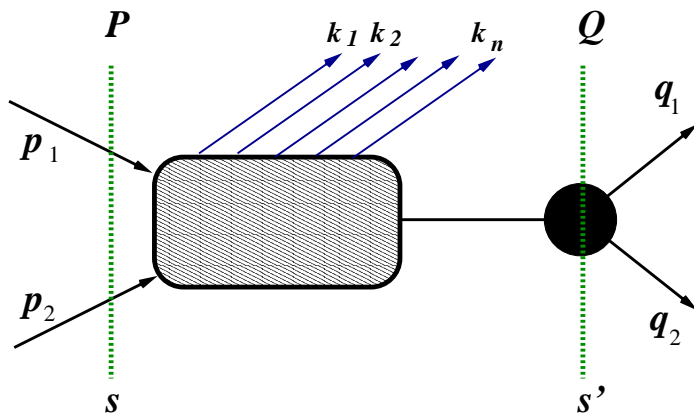
$$\sigma_n = \int ds' \int d\cos\omega d\psi \int \prod_{j=1}^n \frac{d^3 k_j}{2k_j^0} \delta(s - (\hat{Q} + \sum_{j=1}^n k_j)^2) \frac{s'}{2s} \beta(s', q_1^2, q_2^2) \Sigma_n(p_i, q_j, k_l).$$

and

$$k_i|_{CMS} = L_A k_i, \quad q_i|_{CMS} = L_A \hat{q}_i,$$

$$L_A = R_3(\psi) R_2(\omega) B_{\hat{P}P} = R_3(\psi) R_2(\omega) B_{\hat{P}}^{-1},$$

$$\hat{P} = \hat{Q} + \sum_{j=1}^n k_j, \quad \hat{Q} \equiv \hat{q}_1 + \hat{q}_1 = (0, 0, 0, \sqrt{s'})$$



**Note:** Angles  $\omega, \psi$  should not be interpreted as angles of some particle in some frame, they are Euler-type “rigid-body” parametrization the system  $(q_1, q_2)$ .

**Conclusions**

- **Cascade parametrization of the ph.sp. for MC's with multibranching is most common and very succesful.**
- **Lorentz group parametrization is an interesting and very powerfull alteranative (or extension). Keep it in mind!**