Exponentiation in QED: Example of Combining Real and Virtual Corrections

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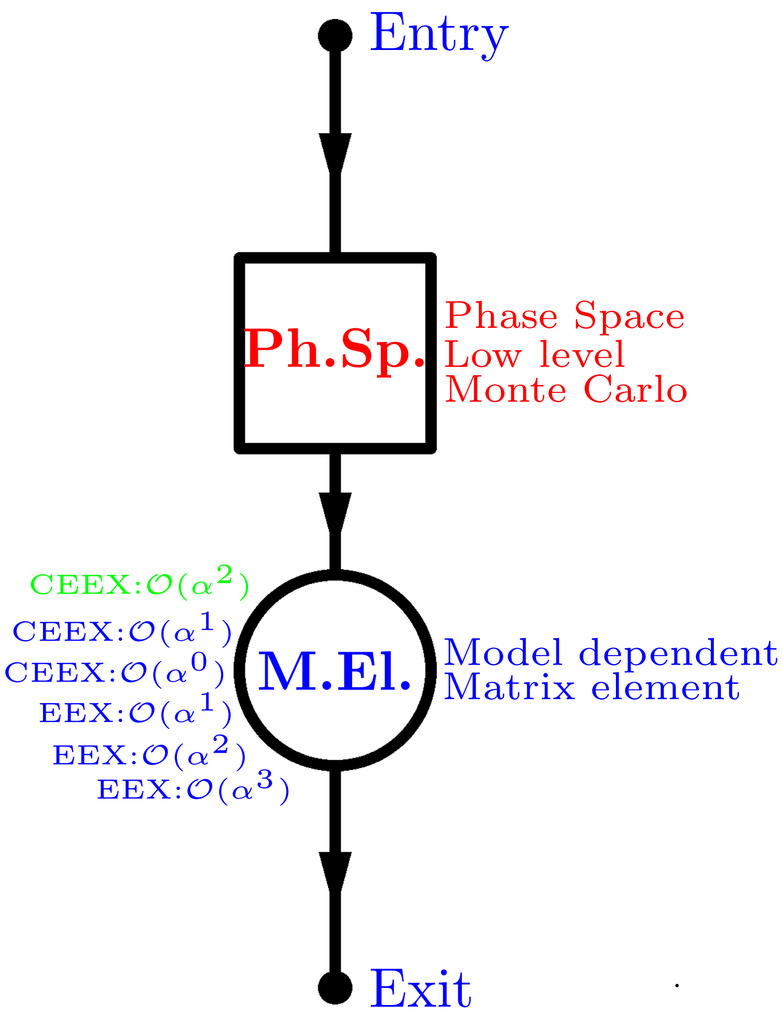
The aim is to summarize briefly on: (1) the past practical experience on combining real and virtual corrections within the exponentiation schemes based on Yennie-Frautschi-Suura method and its extensions, (2) still unexploited freedom in these schemes, concerning combining real and virtual contributions, which may have interesting practical importance for future works.

Related papers by S. Jadach, B.F.L. Ward and Z. Wąs:

Phys. Rev. D **63**, 113009 (2001), Phys. Lett. B449 (1999) 97 and CERN-TH/98-235, EPJ in print.

These and related slides on http://home.cern.ch/jadach

Textbook principle “matrix element × full phase space” ALWAYS ASSUMED



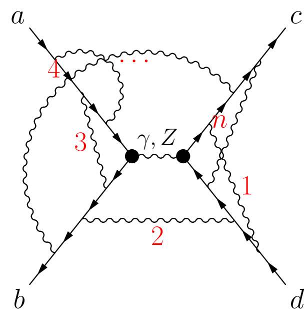
In the Monte Carlo realization it means that:

- Universal Phase-space Monte Carlo simulator is a separate module producing “raw events” (including importance sampling)
- Library of several types of SM/QED matrix elements provides “model weight” is another independent module
- Tau decays and hadronization come after of course.

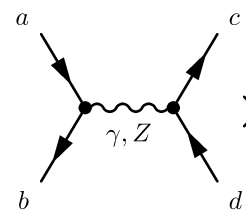
YFS exponentiation in a nutshell

Factorization of virtual IR by Yennie-Frautschi-Suura (1961):

$$\sum_{n=0}^{\infty}$$



$$= e^{\alpha B_4}$$



$$\times (1 + \Delta_{\rm finite})$$

where $B_4(p_a, ..., p_d) = \int \frac{d^4 k}{k^2 - m_\gamma^2 + i\epsilon} \frac{i}{(2\pi)^3} \left| J_I(k) - J_F(k) \right|^2,$

$$J_I = eQ_e\big(\hat{J}_a(k) - \hat{J}_b(k)\big), \quad J_F = eQ_f\big(\hat{J}_c(k) - \hat{J}_d(k)\big), \quad \hat{J}^{\mu}_f(k) \equiv \frac{2p_f^{\mu} + k^{\mu}}{k^2 + 2kp_f + i\epsilon}$$

Factorization of real IR contributions proceeds similarly.

YFS technique of IR resummation is based on Lagrangian, Feynman rules, standard renormalization technique, and exact multiphoton phase space integration. No structure function. No renormalization group. In practical application starting point is always a finite-order perturbative calculation.

It can really provide QED prediction with arbitrarily good physical precision.

NB. YFS used LL technique for approx. phase space integration – there were no good computers in 1961! Some readers think wrongly that LL has an essential role in the YFS exponentiation.

New CEEX and old EEX, exponentiation schemes derived from YFS 1961 work

EEX= Exclusive EXponentiation, very close to original Yennie-Frautschi-Suura 1961

CEEX = Coherent EXclusive exponentiation, is an extension of YFS

Why EXPONENTIATION? Infrared (IR) contributions summed up to $\mathcal{O}(\alpha^\infty)$

In EEX real IR singularities are isolated/reorganized at the **spin amplitude** level while in EEX this is done for differential distributions. Automatic cancellations of IR singularities to $\mathcal{O}(\alpha^\infty)$.

Both EEX and CEEX are based on Lagrangian, Feynman rules and standard renormalization technique. (The use of renormalization group is not required.)

Why EXCLUSIVE? Because Phase-Space $\times \sum_{spin} \left|\mathcal{M}\right|^2$

Contrary to Parton-Showers, Leading-Logs, Structure-Functions – all inherently inclusive.

Why COHERENT? Friendly to Quantum coherence;

Coherence among Feynman diags.: Narrow resonances, $\gamma \oplus Z$ exchanges, $t \oplus s$ channels, ISR$\oplus$FSR etc. Complete $\left| \sum_{diagr.}^{n} \mathcal{M}_i \right|^2$ rather then often incomplete $\sum_{i,j}^{n^2} \mathcal{M}_i \mathcal{M}_j^*$.

EEX/YFS very schematically, ISR $\mathcal{O}(\alpha^1)$ Example:

$$e^-(p_1,\lambda_1) + e^+(p_2,\lambda_2) \to f(q_1,\lambda_1') + \bar{f}(q_2,\lambda_2') + \gamma(k_1,\sigma_1) + \ldots + \gamma(k_n,\sigma_n)$$

$$\sigma = \sum_{n=0}^{\infty} \int_{m_\gamma} d\Phi_{n+2}\; e^{Y(m_\gamma)} D_n(q_1,q_2,k_1,\ldots,k_n)$$

$$D_0 = \bar{\beta}_0$$

$$D_1(k_1) = \bar{\beta}_0 \tilde{S}(k_1) + \bar{\beta}_1(k_1)$$

$$D_2(k_1,k_2) = \bar{\beta}_0 \tilde{S}(k_1)\tilde{S}(k_2) + \bar{\beta}_1(k_1)\tilde{S}(k_2) + \bar{\beta}_1(k_2)\tilde{S}(k_1)$$

$$D_n(k_1,k_2\ldots k_n) = \bar{\beta}_0 \tilde{S}(k_1)\tilde{S}(k_2)\ldots\tilde{S}(k_n) + \bar{\beta}_1(k_1)\tilde{S}(k_2)\tilde{S}(k_3)\ldots\tilde{S}(k_n) + \tilde{S}(k_1)\bar{\beta}_1(k_2)\tilde{S}(k_3)\ldots\tilde{S}(k_n) + \ldots + \tilde{S}(k_1)\tilde{S}(k_2)\tilde{S}(k_2)\ldots\bar{\beta}_1(k_n)$$

Real soft factors: $\tilde{S}(k) = \sum_\sigma |\mathfrak{s}_\sigma(k)|^2 = |\mathfrak{s}_+|^2(k) + |\mathfrak{s}_-(k)|^2 = -\frac{\alpha}{\pi}\left(\frac{q_1}{kq_1} - \frac{q_2}{kq_2}\right)^2$

IR-finite building blocks:

$\bar{\beta}_0 = \left(e^{-2\Re B} \sum_\lambda |\mathfrak{M}_\lambda^{\text{Born+Virt.}}|^2\right)|_{\mathcal{O}(\alpha^1)}$, λ = fermion helicities, σ = photon helicity

$$\bar{\beta}_1(k) = \sum_{\lambda\sigma} |\mathfrak{M}_{\lambda\sigma}^{\text{1-PHOT}}|^2 - \sum_\sigma |\mathfrak{s}_\sigma(k)|^2 \sum_\lambda |\mathfrak{M}_\lambda^{\text{Born}}|^2$$

Everything in terms of $\sum_{spin} |\ldots|^2$!!!!

CEEX schematically, ISR $\mathcal{O}(\alpha^1)$ Example:

$e^-(p_1,\lambda_1)+e^+(p_2,\lambda_2)\to f(q_1,\lambda'_1)+\bar f(q_2,\lambda'_2)+\gamma(k_1,\sigma_1)+\ldots+\gamma(k_n,\sigma_n)$

$$\sigma=\sum_{n=0}^{\infty}\int_{m_\gamma} d\Phi_{n+2}\sum_{\lambda,\sigma_1,\ldots,\sigma_n}\left|e^{\alpha B(m_\gamma)}\mathcal{M}^{\lambda}_{n,\sigma_1,\ldots,\sigma_n}(k_1,\ldots,k_n)\right|^2$$

$\mathcal{M}^{\lambda}_0=\hat\beta^{\lambda}_0$, λ=fermion helicities,

$$\mathcal{M}^{\lambda}_{1,\sigma_1}(k_1)=\hat\beta^{\lambda}_0\mathfrak{s}_{\sigma_1}(k_1)+\hat\beta^{\lambda}_{1,\sigma_1}(k_1)$$

$$\mathcal{M}^{\lambda}_{2,\sigma_1,\sigma_2}(k_1,k_2)=\hat\beta^{\lambda}_0\mathfrak{s}_{\sigma_1}(k_1)\mathfrak{s}_{\sigma_2}(k_2)+\hat\beta^{\lambda}_{1,\sigma_1}(k_1)\mathfrak{s}_{\sigma_2}(k_2)+\hat\beta^{\lambda}_{1,\sigma_2}(k_2)\mathfrak{s}_{\sigma_1}(k_1)$$

$$\mathcal{M}^{\lambda}_{n,\sigma_1,\ldots\sigma_n}(k_1,k_2,\ldots k_n)=\hat\beta^{\lambda}_0\mathfrak{s}_{\sigma_1}(k_1)\mathfrak{s}_{\sigma_2}(k_2)\ldots\mathfrak{s}_{\sigma_n}(k_n)+\hat\beta^{\lambda}_{1,\sigma_1}(k_1)\mathfrak{s}_{\sigma_2}(k_2)\ldots\mathfrak{s}_{\sigma_n}(k_n)$$
$$+\mathfrak{s}_{\sigma_1}(k_1)\hat\beta^{\lambda}_{1,\sigma_2}(k_2)\ldots\mathfrak{s}_{\sigma_n}(k_n)+\ldots+\mathfrak{s}_{\sigma_1}(k_1)\mathfrak{s}_{\sigma_2}(k_2)\ldots\hat\beta^{\lambda}_{1,\sigma_2}(k_2)$$

IR-finite building blocks:

$$\hat\beta^{\lambda}_0=\left(e^{-\alpha B}\mathcal{M}^{\rm Born+Virt.}_{\lambda}\right)\Big|_{\mathcal{O}(\alpha^1)},$$

$$\hat\beta^{\lambda}_{1,\sigma}(k)=\mathcal{M}^{\lambda}_{1,\sigma}(k)-\hat\beta^{\lambda}_0\mathfrak{s}_\sigma(k)$$

Everything in terms of $\mathcal{M}$-spin-amplitudes !!!!

Useful freedom in the choice of the IR regulator in the CEEX scheme

and IR disentanglement of the IR singularities in the CEEX scheme

In the CEEX/YFS scheme the IR extraction and summation of virtual IR singularities is done at the amplitude level, for any number of real, arbitrarily hard photons.

The YFS proof (1961) is based on Feynman diagrams and UV subtraction is assumed to be done beforehand. At this stage, IR singularities are assumed to be regularized somehow, with any possible method, either by theoretically less clean or convenient introduction of the finite photon mass m_γ or by working at $D \neq 4$ dimensions. (The proof does not depend on IR regulator choice.)

After reorganizing IR-divergent terms to $\mathcal{O}(\alpha^\infty)$, the IR cancellations do occur independently in two places:

(a) between the exponential formfactor and the real-photon phase space integral

(b) between the various term inside the well defined IR-finite β-functions.

We have, therefore, a freedom to choose different IR regulators (a) and (b).

Case (a) of IR cancellations: YFS formfactor versus real photon integrals

$$\sigma = \sum_{n=0}^{\infty} \int_{m_\gamma} d\Phi_{n+2} \sum_{\lambda,\sigma_1,\dots,\sigma_n} |e^{\alpha B(m_\gamma)} \mathcal{M}^{\lambda}_{n,\sigma_1,\dots,\sigma_n}(k_1,\dots,k_n)|^2$$

Here, for $\int d^3k$ going into $D \neq 4$ dimensions makes little sense, just unnecessary complication, especially for hard photons.

It is an important drawback of the dimensional regularization is that it works really for virtual corrections and does not make sense for real emission phase space.

For type (a) of IR cancellations we may choose finite photon mass or any other convenient “photon energy cut method” which works at $D = 4$.

NB. In the MC implementations of EEX, CEEX the type (a) IR cancellations work entirely numerically.

$$\sigma = \sum_{n=0}^{\infty} \int_{k_j^0 > \varepsilon\sqrt{s}/2} d\Phi_{n+2}\; e^{Y(\varepsilon)} \sum_{\lambda,\sigma_1,\dots,\sigma_n} |\mathcal{M}^{\lambda}_{n,\sigma_1,\dots,\sigma_n}(k_1,\dots,k_n)|^2$$

where

$$Y(\varepsilon) = 2\alpha\Re B(m_\gamma) + 2\alpha\tilde{B}(m_\gamma) = 2\alpha\Re B(m_\gamma) + \int_{k^0<\varepsilon\sqrt{s}/2} \frac{d^3k}{2k^0}\; \tilde{S}(k, m_\gamma).$$

In the evaluation of $Y(\varepsilon)$ we may use instead of m_γ the dimensional regularization for IR divergences, or any other one. (UV are absent anyway).

NB. Expression for virtual B-function in D-dimensions is given in recent paper of G. Passarino.

Case (b) of IR cancellations: in construction and evaluation of β-functions

Two kinds of IR cancellations present: real-real and virtual-virtual (no real-virtual). The real-real ones occur for the integrand of the real-emission phase space before $\int d^3k$ integration, even better, they occur for the spin amplitudes, before taking square! Example: $\hat{\beta}^{\lambda}_{1,\sigma}(k) = \mathcal{M}^{\lambda}_{1,\sigma}(k) - \hat{\beta}^{\lambda}_{0}\mathfrak{s}_{\sigma}(k)$. They do not require in CEEX any IR regulation and occur in the MC based on CEEX entirely numerically.

For the virtual components of the β's, the numerical execution of the virtual-virtual IR cancellations is in principle possible before $\int d^Dk$ integration for the new methods of calculating the virtual corrections like these of Remiddi et.al. and of Tkachov. In such a case IR-regulator is unnecessary.

$$\hat{\beta}^{\lambda}_{0} = \left(e^{-\alpha B}\mathcal{M}^{\rm Born+Virt.}_{\lambda}\right)\Big|_{\mathcal{O}(\alpha^1)} = \mathcal{M}^{\rm Born+Virt.}_{\lambda} - \alpha B\, \mathcal{M}^{\rm Born}_{\lambda}.$$

In the case of the traditional method of Veltman&Passarino the cancellation of the $1/\epsilon$ terms of IR origin can be done after $\int d^Dk$ integration and executed either analytically or numerically. (Photon mass can also used as a regulator).

All the above is valid for 2 and more loops

Contrast with traditional approach

Note sharp contrast of the class (b) of IR cancellations and the “traditional” Bloch-Nordsieck (BN) method of the killing the IR cancellations in (multi)loop calculations by adding the real photon emission contribution, in the differential x-section (amplitude-squared):

(a) In BN we add IR real photon contribution, while in CEEX we subtract IR virtual,

(b) In BN we do it for more difficult and clumsy differential cross section, while in CEEX we can do it for the spin amplitudes – a much cleaner environment,

(c) and the most important is that in CEEX we may opt for most convenient IR regulator and the most convenient method of subtracting IR virtual part in the multi-loop calculations, independently of the IR regulator method used for the real emissions (handled numerically by the Monte Carlo).

In my opinion there is no better method presently on the market for dealing with IR singularities in QED. Let us hope that it can be extended to collinear singularities and non-abelian IR divergences.

Conclusions

- Yennie-Frautschi-Suura inspired EEX and CEEX methods include built-in mechanism of cancellation of the IR singularities between real and virtual corrections which works well in practice, in many MC applications: KORALZ, BHLUMI, YWSWW3, BHWIDE, KK MC.
- These schemes have still capabilities which may simplify the next generation of the precision exponentiated $\mathcal{O}(\alpha^2)$ calculations (Bhabha for example) and the corresponding Monte Carlo programs.
- Extension to full $\mathcal{O}(\alpha^\infty)$ summation of the collinear singularities is a very desirable extension of CEEX.